THE RÔLE OF THE CONCEPT OF INFINITY IN THE WORK OF LUCRETIUS.

BY PROFESSOR C. J. KEYSER.

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No doubt there will one day be written the history of the concept of infinity. If it is to be done by an American scholar, it will probably not be done in the present generation, for the doing of it calls for a kind of composite scholarly preparation—linguistic, historical, philosophical, scientific, and especially mathematical—which our American universities have indeed the machinery but not yet, it seems, the spirit or the purpose or the atmosphere or the temper to provide.

In any adequate historical survey of the rôle of the notion of infinity in our human thinking a consideration of the De Rerum Natura of Lucretius will have to be accorded the position of an important chapter. Most of the many great merits of the work have been long, if not generally nor even widely, recognized. One of its recognized merits is its superb daring and the unsurpassed magnificence of its enterprise; another is its probably unmatched union of literary excellence with scientific spirit and aim; still another, which includes many, being a highly composite merit, is its confident and often acutely argued presentation, sometimes in detail and sometimes in clear and striking outline, of ideas and doctrines that came into their own only in modern science. I refer to such concepts and dogmas as natural law, the atomic constitution of matter, the conservation of mass and of energy, organic evolution, spontaneous or chance variation of life forms, struggle for existence, survival of the fit, and sensation as the ultimate basis of knowledge and the ultimate test of reality, not to mention other equally brilliant anticipations of modern scientific thought.

In attempts to appraise the work of Lucretius his employment of the notion of infinity is commonly indicated, but only more or less incidentally. For example, in Masson's large volume, Lucretius, Epicurean and Poet, the term infinite has only a subordinate place in the index of important terms; in

Munro's very extensive Notes the term receives but scant attention; and it receives even less in the Notes found in Cyril Bailey's recent and deservedly much praised English translation of the poem. What is missed in such appreciations and commentaries and what I wish to signalize here is the fact that the concept of infinity—of infinite multitude and infinite magnitude—is not merely one among the many ideas, but is indeed the dominant idea, in the work of Lucretius. A critical examination of the work cannot fail to discover that in the author's judgment the concept in question was at once the most powerful of his instruments and the one most obviously indispensable to the success of his great undertaking. That undertaking was a pretty large contract, being nothing less than the endeavor to show forth a method by which it would be possible to explain, or to account for, all phenomena (whether mental or not) without having to resort to the hypothesis of divine intervention.

This is not the place to give a detailed account of the Lucretian principles and procedure. For the purpose of this note it is sufficient to point out that among the fundamental propositions there are three major propositions and that these owe their efficacy and their dominance of the entire discourse to the fact of their postulating the existence of infinite multitude and infinite magnitude. These propositions are that the universe of space is a region or room of infinite extent; that time is an infinite duration composed of an infinite past and an infinite future; and that the matter in the universe is composed of an infinite multitude of absolutely solid (non-porous) and non-decomposable atoms or "seeds of things" always moving hither and thither in an infinite variety of ways and ever so distributed throughout the whole of space that of all spheres none but such as are microscopically minute could at any given instant fail to enclose one or more of the "seeds." Without these postulated infinites explanation of the phenomena of the world was, in the belief of Lucretius, impossible; with them, supplemented by certain other postulates, such explanation was possible. Though the mentioned postulates were not in themselves sufficient, they were regarded as conspicuously necessary. In the view of Lucretius cosmic history was an eternal drama enacted by an infinitude of unoriginated and indestructible elements operating upon an infinite stage. The drama was not to be understood except by help of the

concept of infinity; and so the De Rerum Natura may be not unjustly said to be a kind of poetic celebration of what the author deemed to be the scientific efficacy of that concept.

What did Lucretius mean by infinity? What did he mean by an infinite multitude and by an infinite magnitude? No formal definition of any of these terms is to be found in his work. But it is perfectly clear that he conceived an infinite multitude of elements to be a multitude which could not be exhausted by removing from it one element at a time but which could be thought as arranged in an endless succession of elements. In other words, an infinite multitude signified what we now describe as a denumerably infinite multitude. There is no hint at any conception of any higher order of infinity.

It is noteworthy that, with the possible exception of time, the fundamental infinites of Lucretius were not mere variables capable of increase beyond any prescribed finite amount; on the contrary, they were, like the infinites of Cantor, constant or static affairs; but, unlike the Cantor infinites, those of Lucretius were composed of actual, or concrete, things and not abstract ones like points, for example, or pure numbers; thus the Lucretian infinitude of atoms were material particles and they all existed at once.

Was Lucretius aware of the fact that an infinite multitude, as conceived by him, contained parts or submultitudes equivalent, as we now say, to the whole? He was not only aware of it but he repeatedly employed this characteristic property of infinity effectively and correctly. This rather astonishing fact is sufficiently interesting to justify citation of one or two of the numerous passages supporting its assertion. If we bear in mind that one of the Lucretian infinites was the succession of time units (days or generations or other stretches) beginning at any given instant and together composing what is called the future, the following famous passage makes it perfectly clear that, according to its author, the removal of any finite multitude of elements from an infinite multitude of them leaves a remainder, a part, exactly equal or equivalent to the whole:

"Nor by prolonging life do we take one tittle from the time past in death nor can we fret anything away, whereby we may haply be a less long time in the condition of the dead. Therefore you may complete as many generations as you please;

none the less however will that everlasting death await you; and for no less long a time will he be no more in being, who beginning with today has ended his life, than the man who has died many months and years ago." *

Lucretius, as already said, postulated the existence of an infinity of atoms. These were not all of them identical in shape or in size but in these respects presented a finite number of varieties. Each variety, it was further assumed, constituted an infinite multitude.† These infinites were not distinguished as such; in respect of multiplicity, though not in respect of their physical functions, they were equivalent to one another and to the whole multitude constituted by their elements.

It is sufficiently evident that the poet's conception of infinite multitude was identical with that now employed by mathematicians. Careful scrutiny of the poem will discover that the same may be said of the author's conception of infinite magnitude. Formal definition of the notion is not present. We are told, however, that in respect of size, all the atoms are between a finite upper bound and a finite lower bound; we are told that the sum of any finite number of atoms is finite; we are told that the sum of all atoms of a given form is infinite and that, therefore, the number of them must be infinite. It is thus evident that for Lucretius an infinite magnitude is a magnitude greater than the sum of any finite number of finite quantities none of which surpasses, in respect of parvitude, a finite size.

Formation or possession of ideas is one thing; logically correct handling of them in argumentation is quite another. In his use of the ideas in question Lucretius was sometimes right and sometimes wrong. Of right use some examples have already been given and it would be easy to cite others. As an instance of erroneous use the following passage (as correctly translated by Munro, page 15) may be cited on account of its great interest:

"Again unless there shall be a least, the very smallest bodies will consist of infinite parts, inasmuch as half of a half will always have a half and nothing will set bounds to the division. Therefore between the sum of things and the least of things

* Munro's Lucretius, 4th ed., p. 83.

† Cf. Munro, pp. 39, 40, 41, or Bailey's Lucretius On the Nature of Things, pp. 76, 83, 84.

what difference will there be? There will be no distinction at all; for how absolutely infinite soever the whole sum is, yet the things which are smallest will equally consist of infinite parts."

The significance of the passage and the nature of the error contained in it will be clearer if we note that it is a portion of the argument by which the author seeks to establish his contention that among the parts composing an atom (which is by assumption the smallest material particle capable of existing spatially separate from all other particles) there is a least part, by which is meant a part so small that none of the parts is smaller. Paraphrased in modern terms this portion of the argument would run about as follows: if among the parts composing an atom and being such that no two of them have points in common save points of a common surface there be no least part, then the atom consists of an infinite number of parts; the number of atoms in the universe is infinite; these two infinite multitudes of finite portions of matter are equivalent; the sum of the elements of the latter multitude is an infinite magnitude; such, too, is the sum of the elements of the former multitude; but this sum is the atom itself; hence, unless there be a least part among the parts of an atom, an atom is an infinite magnitude and as such is no less than the sum of all matter. The error is not due to a wrong conception of infinity, whether of multitude or of magnitude, but is obviously due solely to the tacit assumption of the false proposition that the sum of the elements of any infinite multitude of finite elements is infinite. An obvious moral is that a little knowledge of the convergence of series would greatly improve the philosophy of poets and the science of philosophers.

It is astonishing that the mentioned fallacy occurs in immediate conscious connection with a line seeming to refute it: "the half of the half will always have a half." What is the explanation? It is not to be found in any supposition of stupidity or of momentary nodding. It is to be found in the author's purpose and point of view. He was exclusively concerned with natural phenomena, with what he deemed to be existing entities. Regarding the series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$, he would have said in effect: "Composed of man-made symbols like words, it is not nor ever can be endless; to speak of the sum of a non-existing endless series is meaningless; moreover, even if we supposed the series to be endless, to be summable,

and to have 1 for its sum, this 1 would be neither finite nor infinite, for it would not be a magnitude, inasmuch as the summands are themselves not magnitudes but are merely empty abstract symbols; if 1 be said to be a magnitude, in the sense of representing a magnitude, then, if the magnitude 1 be composed of two equal magnitudes, $\frac{1}{2}$ will be a magnitude in the same sense, and so on for the other symbols; if all the symbols be magnitudes in the indicated sense, the summation of the abstract series will be the summation of an endless series of magnitudes; otherwise not; and now what I have contended in my book is that, if the magnitude 1 be finite, not more than a relatively few of the symbols of the series can be magnitudes, and this contention, denying the infinite divisibility of finite magnitude, is based on grounds other than that advanced in the above-quoted passage from my argument." This is not the place to recount, much less to estimate, those "other" grounds. It must, however, be said, in passing, that one of them is in point of kind almost perfectly indicated by the following words of Clerk Maxwell:* "What we assert is that after we have divided a body into a certain finite number of constituent parts called molecules, then any further division of these molecules will deprive them of the properties which give rise to the phenomena observed in the substance."

The classic form of the false thesis tacitly invoked by Lucretius in the above-quoted passage to fortify his "other" grounds for denying the infinite divisibility of a finite magnitude is exceedingly vague: all infinities are equal. It has had an age-long and world-wide vogue. Thus Kanadi, an old Hindoo author, employs the thesis to prove that, if every body be infinitely divisible, there can be "no difference of magnitude between a mustard seed and a mountain." † In this connection mathematicians, especially those who may be interested in the history of the notion of infinity, will be glad to have their attention called to a little-known letter‡ of Newton dealing with the matter. The letter, which is addressed to Richard Bentley, is very interesting on several accounts: it points out the vagueness and the falsity of the above-mentioned thesis, which Dr. Bentley had assumed to be true; it repeatedly employs the term infinite in a sense not less

* Theory of Heat, p. 285.
† See Daubeny's Introduction to the Atomic Theory, p. 5.
‡ See The Works of Richard Bentley, vol. III, p. 207.

vague and indeterminate; it virtually asserts that, if two infinite magnitudes be equal, the addition of any finite magnitude to either of them will destroy the equality.

In closing this note I desire to guard against the danger of leaving a false impression. The mere correctness of the Lucretian concept of infinity by no means accounts for the immense rôle of the concept in the author's work. The secret lies in the fact that the concept so powerfully stimulated the imagination of a great thinker and poet as to cause him to express and to preserve in immortal form a body of ideas which he had acquired from the then still extant works of Epicurus and which after the long lapse of centuries are found to be among the most fruitful scientific ideas of our time.

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ON THE INVARIANT NET OF CUBICS IN THE STEINERIAN TRANSFORMATION.

BY PROFESSOR ARNOLD EMCH.

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1. By Steinerian transformation* we understand an involutorial quadratic Cremona transformation, defined as the one-to-one correspondence between the points of a plane (with the exception of the points of a certain trilateral) and the points of concurrence of their polars with respect to the conics of a pencil. If we use the base points A_1, A_2, A_3, E of the pencil as the vertices and the unit point of a system of projective coordinates, the Steinerian transformation may easily be established in the form

$$\begin{aligned} \rho x_1' &= x_1(x_2 + x_3 - x_1), \\ \rho x_2' &= x_2(x_3 + x_1 - x_2), \\ \rho x_3' &= x_3(x_1 + x_2 - x_3). \end{aligned} \tag{1}$$

The base points $A_1(1, 0, 0)$; $A_2(0, 1, 0)$; $A_3(0, 0, 1)$; $E(1, 1,$

* See *Annals of Mathematics*, vol. 14 (1912), pp. 57–71.

1) of the pencil are invariant under the transformation, while to the diagonal points $E_1(0, 1, 1)$; $E_2(1, 0, 1)$; $E_3(1, 1, 0)$ correspond the opposite diagonals of the diagonal triangle of the base point quadrangle. The net of cubics through the seven points A_1, A_2, A_3, E, E_1, E_2, E_3 may be represented in the form

$$(2) \quad a_1x_2x_3(x_2 - x_3) + a_2x_3x_1(x_3 - x_1) + a_3x_1x_2(x_1 - x_2) = 0,$$

and is invariant under the transformation (1). It is not difficult to establish the

THEOREM: *For every set of values* $(a_1 : a_2 : a_3)$ *equation* (2) *represents a cubic for which the coordinate triangle* $A_1A_2A_3$ *and the unit point* E *form a Steinerian quadruple, such that the tangents to the cubic at these points meet in a point* P *of the cubic whose coordinates are precisely* a_1, a_2, a_3.

Conversely, when from a point P *of an elliptic cubic we draw the four tangents to the cubic with the points of tangency* A_1, A_2, A_3, E, *and we choose these as the vertices and unit point of a projective coordinate triangle, in which the coordinates of* P *are* a_1, a_2, a_3, *then the equation of the cubic may be written in the form* (2).

2. It is known that on every straight line l in a general position there are two points P and P' only which correspond to each other in the Steinerian transformation. They are obtained as the points of intersection of l and the corresponding conic K through the diagonal points, or also as the double points of the involution cut out on l by the conics of the pencil through the base points. The cubic may be generated by the pencil of lines through P and the projective pencil of corresponding conics through E_1, E_2, E_3, and P'. It may also be considered as the locus of pairs of corresponding points on the rays of the pencil through P. For the sake of brevity the point of concurrence of the tangents to a cubic at the points of an inscribed Steinerian quadruple shall be called an S-point of the cubic. Two cubics C_P and C_Q of the net intersect in two points X and X'. As both are invariant under the transformation, the corresponding point of X must lie on C_P as well as on C_Q, and must therefore coincide with X'. Hence the

THEOREM: *Two cubics of the net associated with the complete quadrangle of a Steinerian transformation always intersect in two corresponding points.*

All cubics of the net have, of course, the seven fixed points of the complete quadrangle in common.

3. Let $P(a_1, a_2, a_3)$ and $Q(b_1, b_2, b_3)$ be the S-points of two cubics C_P and C_Q of the net, so that the equation of the pencil of cubics C_λ determined by C_P and C_Q may be written in the form

$$(3)\quad (a_1+\lambda b_1)x_2x_3(x_2-x_3)+(a_2+\lambda b_2)x_3x_1(x_3-x_1) + (a_3+\lambda b_3)x_1x_2(x_1-x_2)=0.$$

Obviously, the S-point of every cubic of this pencil with the parameter λ has the coordinates

$$a_1+\lambda b_1, \qquad a_2+\lambda b_2, \qquad a_3+\lambda b_3,$$

and lies on the line joining the S-points P and Q of C_P and C_Q.

As the equation of every pencil of the net may be written in the form (3) we may state the

THEOREM: *The S-points of the cubics of a pencil within the net of cubics associated with a Steinerian quadruple are collinear, and form a point set projective with the pencil of cubics.*

Let l be the line of collinearity of the S-points of the pencil of cubics, and S_λ the S-point on l associated with the cubic C_λ of the pencil. The cubic C_λ will pass through the two corresponding points P and P' of the Steinerian transformation which lie on l. As there are two and only two such points on l, it is clear that all cubics of the pencil must pass through these points. We shall call the locus l of the S-points of the pencil the S-line of the pencil of cubics C_λ.

Consider now the pencil of cubics C_λ, and join every S-point on the S-line l to a fixed point S not on l. In this manner we get the pencil of lines through S with the equation (s_1, s_2, s_3 being the coordinates of S)

$$(4)\quad (s_2a_3-s_3a_2)x_1+(s_3a_1-s_1a_3)x_2+(s_1a_2-s_2a_1)x_3 + \lambda\{(s_2b_3-s_3b_2)x_1+(s_3b_1-s_1b_3)x_2 + (s_1b_2-s_2b_1)x_3\}=0,$$

which is projective with the set of S-points, and consequently with the pencil of cubics (3). These two projective pencils generate a quartic whose equation is found as

$$(5)\quad \{(a_2b_3 - a_3b_2)x_1 + (a_3b_1 - a_1b_3)x_2 + (a_1b_2 - a_2b_1)x_3\} \times \{s_1x_2x_3(x_2 - x_3) + s_1x_3x_1(x_3 - x_1) + s_3x_1x_2(x_1 - x_2)\} = 0.$$

Clearly, the quartic degenerates into the S-line of the pencil of cubics, and into a cubic having (s_1, s_2, s_3) as the S-point. Hence the

THEOREM: *Any cubic of the net with a given S-point may be generated by any pencil of cubics within the net, not containing the given cubic, and the projective pencil of lines joining the S-point of the given cubic to the S-points of the cubics of the pencil.*

Consider next two pencils of cubics C_λ with the S-line l, and C_μ with the S-line m, and a point S, not on l or m. Draw any line g through S, cutting l and m in S_λ and S_μ, and construct the cubics C_λ and C_μ having S_λ and S_μ as S-points.

They both pass through the two fixed points P and P' on g corresponding to each other in the Steinerian transformation. But P and P' also lie on the cubic C_S associated with S as an S-point. For a variable g through S, S_λ and S_μ describe two perspective point sets on l and m which are projective with the pencils of cubics C_λ and C_μ. These pencils are therefore themselves projective, and generate the cubic C_S. Hence the

THEOREM: *Every cubic of the net associated with a Steinerian transformation may be generated in an infinite number of ways by projective pencils of cubics of the same net.*

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SOME THEOREMS OF COMPARISON AND OSCILLATION.

BY PROFESSOR TOMLINSON FORT.

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THEOREM I: *Given*

$$(1)\quad \frac{d}{dx}(K_1(x)y_1') + G_1(x)y_1 = 0$$

and

$$(2)\quad \frac{d}{dx}(K_2(x)y_2') + G_2(x)y_2 = 0,$$

where K_1, K_2, G_1 and G_2 are real and absolutely integrable in the Riemann sense over the interval $a \leqq x \leqq b$, and where $0 < K_2 \leqq K_1$ and $0 < G_1 < G_2$.

Let y_1 and y_2 be particular, not identically vanishing, solutions of (1) *and* (2) *respectively such that either*

Case I $$y_1(a) = 0$$

or

Case II $$\frac{K_1(a)y_1'(a)}{y_1(a)} \geqq \frac{K_2(a)y_2'(a)}{y_2(a)}.$$

Let $V_i(x) = \alpha_1(x)K_i(x)y_i'(x) + \alpha_2(x)y_i(x)$, $i = 1, 2$, where $\alpha_1(x)$ and $\alpha_2(x)$ are real functions defined and continuous when $a \leqq x \leqq b$ and such that $\alpha_1(x) \neq 0$ at any point and $\alpha_2(x)/\alpha_1(x)$ never decreases as x increases. Assume that $y_1(a) \geqq 0$, $y_2(a) \geqq 0$ and $V_1(a) \cdot V_2(a) \geqq 0$ and that, if $V_2(a) = 0$, $V_1(a) = 0$ also.

If $V_1(x)$ has a root at a point c, where $a < c \leqq b$, then $V_2(x)$ has a root at a point d, where $a < d < c$.

Without loss of generality we assume $\alpha_1 > 0$. If $\alpha_1 < 0$ we simply change the signs of both α_1 and α_2.

Case I. Assume $y_1'(a) > 0$ and, if $y_2(a) = 0$, $y_2'(a) > 0$ also, in which again there is no loss of generality. Let $l = a + \delta$ where $\delta > 0$ is very small. Then

$$K_1y_1'y_2 - K_2y_2'y_1]_a^l = \int_a^l (G_2 - G_1)y_1y_2dx + \int_a^l (K_1 - K_2)y_1'y_2'dx.$$

In case $y_2(a) = 0$ and δ is sufficiently small, $y_1y_2 > 0$ when $a < x \leqq l$ and $y_1'y_2' > 0$ when $a \leqq x \leqq l$. Consequently

$$\frac{K_1(l)y_1'(l)}{y_1(l)} > \frac{K_2(l)y_2'(l)}{y_2(l)}.$$

If $y_2(a) \neq 0$ this result is immediate from the continuity of y_1, y_1', y_2 and y_2'. Replace the point a by the point l and Case I reduces to Case II.

Case II. Assume that $V_2(x)$ does not have a root on the interval $a < x < c$ and assume that c is the first root of $V_1(x)$ greater than a. Choose $y_1(a) = y_2(a)$.

(*a*) Suppose $V_2(a) > 0$ and consequently $V_1(a) > 0$.

$$y_1^2\left(\frac{K_1y_1'}{y_1}-\frac{K_2y_2'}{y_2}\right)\Big]_a^c=\int_a^c(G_2-G_1)y_2^2dx$$
$$+\int_a^c(K_1-K_2)y_2'^2dx+\int_a^c K_2\left(\frac{y_1'y_2-y_1y_2'}{y_2}\right)^2dx.*$$

y_1 and y_2 are both positive over the interval of integration since

$$\frac{K_1(x)y_1'(x)}{y_1(x)}$$

becomes negatively infinite as x approaches c, and similarly

$$\frac{K_2(x)y_2'(x)}{y_2(x)}$$

becomes negatively infinite as x approaches the first root of y_2 greater than a. Consequently

$$K_1(c)y_1'(c)y_2(c)-K_2(c)y_2'(c)y_1(c)>0.$$

Combining this with $V_1(c)=0$, we get $y_1(c)V_2(c)<0$, which is impossible.

(*b*) $V_2(a)\leqq 0$ and $V_1(a)\leqq 0$.

$$\frac{K_i(x)y_i'(x)}{y_i(x)} \qquad (i=1,2)$$

decreases as x increases, as is proved by differentiation. Consequently $y_i(x)=0$ to the right of a necessarily before $V_i(x)=0$. Suppose $y_1=0$ at g, where $g<c$; then, as is well known, $y_2=0$ at h where $h<g$. There is only one such point less than c, for by hypothesis $V_2(x)$ has no root when $a<x<c$. Take a point $l=g+\delta$, where δ is small. At this point, with δ small enough, we have the same situation as at a under (*a*) if y_2 is replaced by y_2 multiplied by such a constant as to cause $y_2(l)$ to equal $y_1(l)$ and then y_1 and y_2 replaced by $-y_1$ and $-y_2$ respectively.

THEOREM II. *Let y_1 and y_2 be any two not identically vanishing solutions of* (1) *and* (2) *respectively and let*

$$V_i(x)=\alpha_1(x)K_i(x)y_i'(x)+\alpha_2(x)y_i(x) \qquad (i=1,2).$$

Between any two successive roots of $V_1(x)$ on the interval $a\leqq x\leqq b$ lies at least one root of $V_2(x)$.

* This is a well known identity. It can easily be verified.

Suppose two successive roots of $V_1(x)$ at c_1 and c_2 respectively. Without loss of generality we assume $y_1(c_1) > 0$ and $y_2(c_1) \geqq 0$; if $y_2(c_1) = 0$, $y_2'(c_1) > 0$. Since in addition $K_1(x)y_1'(x)/y_1(x)$ decreases in x and $\alpha_1 > 0$, $V_1(x) < 0$ when $c_1 < x < c_2$.

Case I. $V_2(x) < 0$ when $c_1 < x < c_2$. Conditions at a of Theorem I, Case II b, are fulfilled at c_1.

Case II. $V_2(x) > 0$ when $c_1 < x < c_2$, $y_1 = 0$ at g, where $c_1 < g < c_2$, since $K_1(x)y_1'(x)/y_1(x)$ always decreases. At g we have the conditions of Theorem I, Case I, y_1 being replaced by $-y_1$.

Suppose next that we have the equation

$$\frac{d}{dx}(K(x, \lambda)y') + G(x, \lambda)y(x) = 0, \tag{3}$$

where $K(x, \lambda)$ and $G(x, \lambda)$ are real and continuous in λ, $-\infty < \lambda < \infty$, for any x, and with respect to x absolutely integrable in the Riemann sense from a to b, and where $K > 0$ never decreases as λ increases, and G actually increases from values less than or equal to zero to values as large as desired and is such that, for a particular λ, $G \equiv 0$ in x or does not vanish at all.

Consider (3) subject to the boundary conditions

$$\begin{aligned} \beta_1 K(a, \lambda)y'(a) + \beta_2 y(a) &= 0, \\ \alpha_1 K(b, \lambda)y'(b) + \alpha_2 y(b) &= 0, \end{aligned} \tag{4}$$

where α_1, α_2, β_1 and β_2 are real constants, $\alpha_1 > 0$ and β_1 and β_2 are not both zero.

Let $V(x) = \alpha_1 K(x, \lambda)y'(x) + \alpha_2 y(x)$. It is to kept in mind that from now on α_1 and α_2 are constants.

Define $K(x, \lambda)$ and $G(x, \lambda)$, when $x > b$, thus: $K(x, \lambda) = K(x, b)$ and $G(x, \lambda) = G(x, b)$.

Let y, not identically equal to zero, satisfy the first of equations (4). Then notice that V and V' are continuous in λ and that they do not simultaneously vanish when $G \geqq 0$.

Moreover when $G \equiv 0$ and $V(x) \not\equiv 0$, $V(x)$ has at most one root on the interval $a \leqq x \leqq b$, as is readily proved by solving the equation. Consequently in this case when $G > 0$ but sufficiently small, $V(x)$ has only one root on the interval $a \leqq x \leqq b$. $V \equiv 0$ is only possible when $V = 0$ reduces to $y' = 0$. For values of $G > 0$ but sufficiently small in this instance, $V \neq 0$ at all on the interval $a < x \leqq b$, since $K(x, \lambda)y'(x)/y(x)$

decreases as λ increases. Moreover when λ is sufficiently large, as is well known, y and consequently V have as many roots as desired when $a < x < b$. Using these facts and Theorems I and II we now readily prove the following:

THEOREM III. *There exist unique values,* $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, *such that when* $\lambda = \lambda_j$, $j = 1, 2, 3, \cdots$, *a solution* y *of* (3) *exists satisfying* (4) *such that* $V(x)$ *has exactly* j *roots on the interval* $a < x < b$.

We can extend this to $j = 0$ if, when $G = 0$, V has no root on the interval $a < x < b$.

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NOTE ON INFINITE SYSTEMS OF LINEAR EQUATIONS.

BY DR. W. L. HART.

(Read before the American Mathematical Society April 28, 1917.)

IN considering infinite systems of linear equations

$$\sum_{j=1}^{\infty} a_{ij}x_j = x_i' \qquad (i = 1, 2, \cdots), \tag{1}$$

particular interest is attached to those whose solutions preserve the properties of the solutions of a set of n linear equations in n variables. It is known* that the system (1) possesses this property if $a_{ij} = d_{ij} - b_{ij}$ ($d_{ii} = 1$; $d_{ij} = 0$, $i \neq j$), where the infinite matrix $B = (b_{ij})_{i,\, j=1,\, 2,\, \ldots}$ is completely continuous.† The discussion of Riess deals only with the special case $a_{ij} = d_{ij} - b_{ij}$, but it is easily found that his proof holds for the more general case stated below. The proof of the theorem of this note is not given since it differs only in minor details from the proof of the theorem given by Riess.

It will be said that a matrix A_1 is a sub-matrix of the matrix

* Cf. F. Riess, Equations Linéaires, p. 94.
† Cf. F. Riess, loc. cit.

$A = (a_{ij})_{i,\ j=1,\ 2,\ \ldots}$ in case it is obtained from A by striking out certain k rows and k columns (k finite).

Let the system (1) be taken in the field of complex numbers and suppose that the sets of numbers $x = (x_1, x_2, \cdots)$ and $x' = (x_1', x_2', \cdots)$ represent points in Hilbert space; that is, $\sum_{i=1}^{\infty} |x_i|^2$ and $\sum_{i=1}^{\infty} |x_i'|^2$ converge. The main result of the present note is

THEOREM I. *Let the matrix $A = (a_{ij})_{i,\ j=1,\ 2,\ \ldots}$ be limited in the sense of Hilbert.* Suppose there exists a sub-matrix A_1 possessing a unique limited reciprocal A_1^{-1}. Then, the system of linear equations* (1) *preserves the characteristics of a system of n linear equations in n unknowns.*

That is, either (a) for every x' system (1) possesses a unique solution, or (b) there is an integer $h > 0$ such that if x' is a point in Hilbert space satisfying certain h linear relations, there exists a solution of (1) depending linearly on h parameters.

With the help of certain details necessary for the proof of Theorem I, there is easily proved

THEOREM II. *If the matrix A satisfies the hypotheses of Theorem I and if the homogeneous system*

$$\sum_{j=1}^{\infty} a_{ij}x_j = 0 \qquad (i = 1, 2, \cdots) \tag{2}$$

has no solution in Hilbert space except $x = 0$, then A possesses a unique limited reciprocal matrix.

The proof is simple because the reciprocal matrix of the theorem is actually constructed during the proof of Theorem I.

For the system (1) with $a_{ij} = d_{ij} - b_{ij}$ where $B = (b_{ij})$ is completely continuous, the hypothesis of Theorem I is always satisfied by taking, as the sub-matrix A_1, the infinite matrix obtained by striking out the first n rows and n columns where n is taken sufficiently large. Such a matrix is known† to have a unique limited reciprocal matrix.

83RD ARTILLERY,
FORT D. A. RUSSELL, WYOMING,
November 10, 1917.

* Hellinger and Toeplitz, *Math. Annalen*, vol. 69 (1910), p. 299.
† Riess, loc. cit., p. 98.

AN UNDERVALUED KIRKMAN PAPER.

BY PROFESSOR LOUISE D. CUMMINGS.

(Read before the American Mathematical Society October 27, 1917.)

THE purpose of this note is to emphasize the importance in the theory of triad systems of a Kirkman* paper, which appears to have been overlooked by all writers on this subject up to the present time.

A short explanation of the symbols employed in the paper is necessary. The symbol $Q_{x,y,z}$ denotes the greatest number of combinations that can be made with x elements, y at a time, so that no combination of z elements shall be twice employed; for brevity $Q_{x,3,2}$ is replaced by Q_x. The symbol V_x denotes the number of pairs possible with x elements that are excluded from Q_x. The symbol q_x denotes the number of triads formed with x elements, in which no duad is twice employed, q_x being not necessarily a maximum; v_x is the number of duads possible with x elements not employed in q_x. Four pairs such as 12, 23, 34, 41 forming a closed circle are denoted by the symbol C_4.

The object of Kirkman's paper is to determine the value of Q_x, and to establish the following theorems:

$$Q_x = \frac{x(x-1)}{6} - \frac{V_x}{3},$$

where

$$V_x = \frac{x}{2} + 3k + 1 \text{ if } x = 6n - 2;$$

$$V_x = 6k + 4 \quad \text{“ } x = 6n - 1;$$

$$V_x = \frac{x}{2} \quad \text{“ } x = 6n, 6n + 2;$$

$$V_x = 0 \quad \text{“ } x = 6n + 1, 6n + 3;$$

$$2^m(2k + 1) = n;\ n, m, x, k \text{ are integers} \geqq 0.$$

The case of special interest to us is that in which $V_x = 0$,

* *Cambridge and Dublin Math. Journal*, vol. 2 (1847).

for only under this condition does a triad system on x elements exist. Kirkman proves the two following propositions:

I. If $x = 2n + 1$ and $V_x = 0$, then $V_{2x+1} = 0$, and $v_{2x-1} = C_{2x-2}$; that is, if x is odd and a triad system on x elements exists, then a triad system on $2x + 1$ elements may be constructed which contains a circle of $2x - 2$ pairs chosen from $2x - 1$ elements. This theorem gives the construction for one type of the headed triad systems $\Delta_7, \Delta_{15}, \Delta_{19}, \Delta_{27}, \cdots$.

II. If $x = 2n$ and $v_{x+1} = C_x$, then $V_{2x+1} = 0$, and

$$v_{2x-1} = C_{2x-2};$$

that is, if x is even and if from $x + 1$ elements a partial triad system can be formed such that the unemployed pairs form a closed circle of x pairs, then a triad system on $2x + 1$ elements may be constructed which contains a circle of $2x - 2$ pairs chosen from $2x - 1$ elements. This theorem gives the construction for one type of triad systems $\Delta_9, \Delta_{13}, \Delta_{25}, \cdots$.

These two propositions enable Kirkman to prove that a triad system exists for every value of x of the form $6n + 1$ or $6n + 3$. This result has, in general, been attributed to Reiss,* but, as the Reiss paper was not published until 1859, twelve years after that of Kirkman, the credit properly belongs to Kirkman. A comparison of the Reiss paper with that of Kirkman reveals a remarkable identity of method in the work of these two men. The construction given by Reiss for a triad system on $2x + 1$ elements, x being an odd number, is a perfect duplication of the work of Kirkman. The Reiss construction for a triad system on $2x + 1$ elements, x being an even number, may seem at first sight to differ from that given by Kirkman, but investigation reveals the fact that the two methods are essentially the same and always furnish congruent triad systems. This curious duplication by Reiss of the Kirkman work gives rise to the question as to whether Reiss had a knowledge of this Kirkman paper, or whether the problem was of such a type that two investigators would inevitably have followed the same line of argument.

In Crelle's *Journal für Mathematik*, vol. 45 (1853), Steiner proposes some problems in combinations now generally known as the Steiner combination problems. The first of these problems is only a special case of the Kirkman symbol $Q_{x,y,z}$,

*M. Reiss, "Ueber eine Steinersche kombinatorische Aufgabe." *Crelle's Journal*, vol. 56 (1859), p. 326.

and the remaining problems are slight modifications of the Kirkman problem. Since Kirkman published the general problem ($Q_{x,y,z}$) as early as 1844 in the *Ladies' and Gentlemen's Diary*, and, by 1850, had already contributed two papers on the subject to the *Cambridge and Dublin Mathematical Journal*, the *Kirkman* combination problem would seem to be a more appropriate name for the so-called Steiner problem (*a*). Moreover, the Steiner problems (*b*), (*c*), (*d*), (*e*), should be designated as the Steiner modifications of the Kirkman combination problem.

Netto makes no mention of this Kirkman paper either in his Vorlesungen or in his article on Tripelsysteme in the Encyklopädie der Mathematischen Wissenschaften. In the Encyklopädie article, Netto refers only to two papers by Kirkman, one, in volume 7 (1852), and the other, in volume 8 (1853) of the *Cambridge and Dublin Mathematical Journal*. Now, volume 7 contains no paper by Kirkman on any subject, and the paper in volume 8 is Kirkman's third paper on the problem $Q_{x,y,z}$. That Netto in preparing an article for the Encyklopädie should have overlooked entirely Kirkman's two earlier papers, published in the *Journal* in volumes 2 and 5 respectively, seems the more astounding, since he had at his disposal, not only a very good library at Giessen, but also all of the great mathematical libraries of Germany. The comparative rarity of the *Cambridge and Dublin Journal* is the apparent excuse for the ignorance of Kirkman's work and the exaltation of Steiner and Reiss, but this is a weak argument for Netto when he undertakes an Encyklopädie article.

Another point worthy of attention in the Kirkman paper is represented in the notation $v_{2x-1} = C_{2x-2}$. Interpreted in terms of the special case of fifteen elements, Kirkman shows that a triad system constructed by his method contains a closed circle of twelve pairs involving thirteen elements. Here, then, is the first example of what Professor F. N. Cole, in his work on triad systems, has designated as the dodekad interlacing of the triads. Investigation shows that this Kirkman system contains 56 of these dodekad interlacings, and that among the 80 non-congruent triad systems on fifteen elements enumerated by Professor Cole, 78 contain interlacings of this type.

The remaining propositions in this Kirkman paper concern the determination of Q_x when a triad system on x elements

does not exist, and, therefore, are steps towards the solution of the hitherto so-called Steiner problem (*a*).

The school-girl problem is merely an example which originated in the development of this paper on combinations, and Kirkman justly complained of the almost total eclipse of this paper in the wide popular interest aroused by the school-girl problem. The eclipse appears to have continued up to the present day, since no mention is made of this Kirkman paper by Steiner, Reiss, Netto, or by any of the recent writers on triad systems.

VASSAR COLLEGE,
October, 1917.

PIERRE LAURENT WANTZEL.

BY PROFESSOR FLORIAN CAJORI.

(Read before the American Mathematical Society September 4, 1917.)

EVERY one knows that one of the noted proofs of the impossibility of an algebraic solution of the general quintic equation is due to Wantzel. Nevertheless histories of mathematics and biographical dictionaries are silent regarding his life. The eleven papers listed in Poggendorff's Handwörterbuch as due to "Pierre Laurent Wantzell" do not include the proof in question, and a query is raised in a footnote regarding another "Wantzell"; but nowhere does Poggendorff refer to a "Wantzel." Text-books on algebra and the theory of equations do not give Wantzel's full name. The reader is thus left without positive information as to the author of "Wantzel's proof." His name suggests German nationality, as does the name of "Mannheim," of slide-rule fame. Yet both these men were born in Paris and passed their lives at the Polytechnic School there.* Born in 1814, Wantzel died prematurely in 1848. He is the "Pierre Laurent Wantzell" of Poggendorff but in his published articles his name is always spelled

* On the life of Wantzel, see Barré de Saint-Venant in *Nouvelles Annales de Mathématiques* (Terquem et Gerono), vol. 7 (1848), pp. 321–331; A. de Lapparent in Ecole polytechnique, Livre du Centenaire, 1794–1894, vol. I., Paris, 1895, pp. 133–135, see also pp. 63–65, 190; Gaston Pinet's Ecrivains et Penseurs Polytechniciens, 2e éd., Paris, 1902, p. 20; Charles Sturm in *Comptes rendus hebdomadaires des Séances de l'Académie des Sciences*, Paris, vol. 28 (1849), pp. 66, 67.

"Wantzel." Pinet says of him: "Endowed with extreme vivacity of impressions and with truly universal aptitudes, he carried off the prize for a French dissertation and a Latin dissertation at a general competition and, the year following (1832), entered with first rank the Polytechnic School—a double success before unheard of. He studied with inveterate zeal German and Scotch philosophers; he threw himself into mathematics, philosophy, history, music, and into controversy, exhibiting everywhere equal superiority of mind." He became élève-ingénieur des Ponts et Chaussées, then ingénieur; he was appointed répétiteur about the time when such positions were held by Comte, Transon, Bertrand, Bonnet, Catalan, Leverrier, and Delaunay. In 1843 he became examinateur d'admission. Saint-Venant says of him: "He was blameworthy for having been too rebellious to the counsels of prudence and of friendship. Ordinarily he worked evenings, not lying down until late; then he read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours until he was married. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death."

The Royal Society Catalogue of Scientific Papers quotes the titles of 18 papers by Wantzel, and of three more which he brought out jointly with Saint-Venant.

General Quintic Insolvable by Radicals.

As previously stated, Wantzel's most noted scientific achievement is found in his paper "De l'impossibilité de résoudre toutes les équations algébriques avec des radicaux" in the *Nouvelles Annales de Mathématiques*, volume 4 (1845), pages 57–65. The second part of this proof, which involves substitution theory, is reproduced in Serret's Algèbre supérieure.*

At the beginning of his article, Wantzel expresses himself on the proofs of Abel and Ruffini as follows: "Although his (Abel's) demonstration is at bottom exact, it is presented in a form too complicated and so vague that it is not generally accepted. Many years previous, Ruffini, an Italian geometer,

* In the fifth edition, 1885, the part proof occurs in vol. II, pp. 512–517.

had treated the same question in a manner much vaguer still and with insufficient developments, although he had returned to this subject many times. In meditating on the researches of these two geometers and with the aid of principles which we established before,* we have arrived at a form of proof which appears so strict as to remove all doubt on this important part of the theory of equations." His previous paper, to which reference is made, deals with algebraic incommensurables. Wantzel says that the main theorem set forth therein had been previously established by Abel and again by Liouville. Thus, Wantzel expresses special indebtedness to Abel and Liouville, while to Ruffini he makes acknowledgment only in a general way as indicated in the passage quoted above. A different estimate of indebtedness has been made more recently by H. Burkhardt† and E. Bortolotti.‡ They claim that the proof which goes by the name of Wantzel is essentially Ruffini's proof of 1813. Inasmuch as Wantzel's proof has been generally accepted as altogether valid, the implication is that Ruffini's proof is equally valid.

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* *Nouvelles Annales* (Terquem), vol. 2 (1843), pp. 117–127.

† H. Burkhardt, "Die Anfänge der Gruppentheorie und Paolo Ruffini," *Zeitsch. f. Mathematik u. Physik*, 37. Jahrg., Supplement, Leipzig, 1892, pp. 119–159. Burkhardt says (p. 156): "Es braucht wohl kaum noch ausdrücklich hervorgehoben zu werden, dass diese Fassung des Unauflösbarkeitsbeweises sich in allen wesentlichen Punkten mit derjenigen deckt, welche als 'Wantzel'sche Modification des Abel'schen Beweises' in den Lehrbüchern mitgeteilt zu werden pflegt." Again he says (p. 159), "diesen Beweis hat er (Ruffini) nicht nur zuerst durchgeführt, sondern ihn auch nach verschiedenen Umarbeitungen auf die einfache Form gebracht, welche Wantzel zugeschrieben zu werden pflegt."

‡ In the Carteggio di Paolo Ruffini con alcuni scienziati del suo tempo, Roma, 1906, p. 15 (303), Bortolotti says in a footnote that Ruffini's Riflessioni intorno alla soluzione delle equazioni algebraiche generali, Modena, 1813, "contiene la dimostrazione, che per lungo tempo attribuita al Wantzel, fu sempre considerata come la più facile e la più convincente del teorema di Ruffini." Bortolotti expresses himself more fully in his discourse, Influenza dell' Opera Matematica di Paolo Ruffini (Modena), 1902, p. 42: "E sarebbe anche strano che egli (Abel) accusasse la dimostrazione di Ruffini di esser troppo complicata; laddove essa è tanto semplice che, quando il *Wantzel* volle ridurre il teorema a forma facile e piana, fu costretto a riprodurre, nelle sue linee generali, la redazione del Ruffini. L' ingiustizia delle umane cose ha però voluto, e vuole tuttora, che la dimostrazione che il Ruffini ha data del teorema da lui stesso scoperto, sia stata battezzata col nome di *Dimostrazione di Wantzel del Teorema di Abel*. Quella disgraziata denominazione, introdotta dal Serret nel suo rinomato libro di *Algèbre supérieure* è stata ricopiata di testo in testo fino ai nostri giorni: e si trova anche nell' ultimo e più reputato testo di Analisi, quello del Picard, uscito nel 1896 quattro anni dopo la piena ed irrefutabile dimostrazione data dal Burkhardt della priorità del Ruffini."

The question arises, is Wantzel's proof really the same as Ruffini's of 1813? Before answering this query, the question must be considered, how much must be included as constituting an integral part of the proof? Is only the portion involving substitution theory to be taken into account, or is the introductory part relating to irrational algebraic functions and the general expression for the root form of an algebraically solvable equation to be considered as well? Burkhardt and Bortolotti seemingly look upon the application of substitution theory as the only essential part of the demonstration. But, as Netto says, the "proof of the impossibility of an algebraic solution of general equations above the fourth degree can never be obtained from the theory of substitutions alone." *

Moreover the state of algebra at the beginning of the nineteenth century required that both parts be investigated, to establish the impossibility. With Ruffini and Abel, and also with Wantzel, the establishment of the general root form of algebraically solvable equations was the most difficult and serious part which occupied the larger portion of the space devoted to the proofs. Both parts received consideration by William Rowan Hamilton,† L. Kronecker,‡ O. Hölder,§ and J. Pierpont.|| Sylow¶ and Hölder point out that there is an unproved assumption in the *first part* of Ruffini's proof. Neither Ruffini nor Abel were ever seriously criticized for lack of rigor in the strictly substitution theory part, but both were criticized on the other parts. Abel committed an error in the statement of the theorem relating to an algebraic function of the μth order and mth degree, which was corrected by L. Königsberger in 1869.** This error had disturbed William Rowan Hamilton, who, in his long article†† "On the argument of Abel," declared that the error "renders it difficult to judge of the validity of his (Abel's) subsequent reasoning." However, J. Pierpont‡‡ rightly remarks that this slip "does not

* E. Netto, Theory of Substitutions, translated by F. N. Cole, Ann Arbor, 1892, p. 240.

† W. R. Hamilton, "On the argument of Abel," *Transactions of the Royal Irish Society of the year 1839*, p. 248.

‡ L. Kronecker, *Monatsbericht d. k. p. Akademie d. Wiss. zu Berlin*, 1879, p. 205.

§ Encyklopädie d. math. Wissensch., 1. Band, p. 504.

|| J. Pierpont in *Monatshefte f. Math. u. Physik*, VI (1895), pp. 37–51.

¶ Oeuvres of Abel, edition by Sylow and Lie, II, 1881, p. 293.

** L. Königsberger, *Math. Annalen*, vol. 1 (1869), pp. 168, 169.

†† W. R. Hamilton, loc. cit., p. 248.

‡‡ J. Pierpont, loc. cit., p. 47.

affect in the least the validity of the proof." Abel establishes the fundamental theorem, first enunciated by Vandermonde: "If an equation is algebraically solvable, then one can always give its root such a form that all algebraic functions of which it is composed, can be expressed by rational functions of the roots of the given equation." This theorem is fundamental also in Ruffini's research but he failed to give it convincing proof. It is to Abel's credit to have given it a binding demonstration. Abel gives a classification of algebraic functions involving radicals. Wantzel follows Abel, but gives a shorter treatment containing only what is needed for his immediate purpose. In his paper of 1843, Wantzel deals with the properties and classification of radicals and of rational functions of radicals that involve root extractions of real but not of imaginary "quantities." In his paper of 1845, in which the subject is continued and the Ruffini-Abel theorem is established, he states that the results previously proved for radicaux numériques apply equally to radicaux algébriques and proceeds on this assumption. Wantzel's treatment of this first part of his proof is not always happy; Serret in his Algèbre supérieure prefers to adhere to Abel's exposition. There is nothing in Ruffini that directly suggested to Wantzel his mode of classifying algebraic functions. On the other hand, Wantzel's indebtedness to Abel is strongly evident. Nor does Wantzel attempt to conceal this fact. Besides the general statement of indebtedness to which we referred earlier, Wantzel makes special reference to Abel in two places: one where he says that his names for the different classes of radicals are pretty nearly the same as Abel's; the other where he states that his fundamental theorem on radicals was proved before him by Abel and Liouville.*

The Ruffini publication of 1813, previously referred to, consists of viii + 140 pages. The impossibility of an algebraic solution of the general equation of a degree higher than the fourth is given in Part I; Part II treats of the impossibility of solution by the aid of certain transcendental expressions. Here again, a comparatively small portion concerns itself with the part of the proof involving substitution theory. The larger part of the book is devoted to the study of the properties of algebraic functions and of a certain class of tran-

* See Wantzel in *Nouvelles Annales de Mathématiques*, Tome II, Paris, 1843, pp. 125, 127.

scendental functions (espressione trascendente esatta) which, among others, includes trigonometric and logarithmic functions, but is not sharply defined; nor does the many-valuedness of these transcendental functions receive adequate attention. The proof of the impossibility of solution by an espressione trascendente esatta possesses therefore little value.

If, for the moment, we leave out of consideration everything that Ruffini says about transcendental functions, and confine ourselves to his proof of the impossibility of an *algebraic* solution, we find that, although inconclusive in some important details, it must be admitted to his everlasting credit that the general outline of his demonstration is correct. The part of the proof regarding the form which an algebraic root must take (if such an algebraic root exist) is the weak part. Certain theorems necessary for his proof are not established, or rest upon illusory arguments. Letting P be a rational function of the coefficients of the given equation, he introduces the first radical Q by the relation $Q^p = P$ (p prime), then he lets F_1 be a rational function of P and Q, and defines the second radical R by the relation $R^q = F_1$ (q prime), etc. His general root form is $x_n = F$ (P, Q, R, S, etc.), where (page 12) F is a rational function of P, Q, R, S, $\cdots$. It is *assumed* by Ruffini that if the equation can be solved by algebra, the form of its roots must be the one given here.

The part of Ruffini's proof involving substitution theory is free of fault and is in outline as follows:

(1) Let P be a symmetric function of the roots $x_1, x_2, \cdots, x_5$ and unaltered by the cyclic substitution $s = (12345)$. Applying $s, s^2, \cdots$ to y_1, where $y_1^p = P$, gives $y_1^p = y_2^p = \cdots = y_5^p$, and $y_2 = \beta y_1, \cdots, y_5 = \beta^4 y_1$, $y_1 = \beta^5 y_1$; hence $\beta^5 = 1$.

(2) P is unaltered by $(123) = \sigma$. Applying $\sigma, \sigma^2, \cdots$ to y_1, we get by the above process $y_a = \gamma y_1$, $y_{a+1} = \gamma^2 y_1$, $y_1 = \gamma^3 y_1$ and $\gamma^3 = 1$,

(3) The substitutions $(12345)(123) = \tau, \tau^2, \cdots$ applied to y_1 give similarly $\beta^5\gamma^5 = 1$. Hence $\gamma = 1$, and $y_1 = y_a = y_{a+1}$.

(4) The substitutions $(345) = \phi, \phi^2, \cdots$ applied to y_1 give similarly $y_c = \delta y_1$, $y_{c+1} = \delta^2 y_1$, $y_1 = \delta^3 y_1$. Hence $\delta^3 = 1$.

(5) The substitutions $(12345)(345) = \pi, \pi^2, \cdots$ applied to y_1 yield $\beta^5\delta^5 = 1$, hence $\delta = 1$ and $y_1 = y_c = y_{c+1}$. Hence $y_1 = y_a = y_c$. Applying σ to these yields $y_1 = y_a = y_2$, hence $\beta = 1$ and Q is unaltered by $s, s^2, \cdots$. The same conclusion can be reached for the algebraic irrationals R, S, $\cdots$, and

finally for F. Hence $x_n = F(P, Q, R, \cdots)$ is absurd, since its right member is unaltered by certain substitutions while the left member is altered.

The part of Wantzel's proof involving substitution theory is as follows:

(1) Let p be a symmetric function of the roots $x_1, x_2, \cdots x_5$, and y a rational function of the roots, and $y^n = p$. The transposition (12) yields $y_1 = \alpha y$, $y = \alpha y_1$; hence $\alpha^2 = 1$ and $n = 2$; the first radical appearing in the root form is a square root.

(2) In $z^n = p_1$ let z be a rational function of the roots which is altered by a cyclic substitution of three letters, but p_1 is unaltered by it. We obtain $z_1 = \alpha z$, $z_2 = \alpha z_1$, $z = \alpha z_2$; hence $\alpha^3 = 1$ and $n = 3$.

(3) Applying to this same z a cyclic substitution of five letters, he proves that $\alpha^5 = 1$. But $\alpha^3 = 1$; hence $\alpha = 1$.

(4) Hence when the degree of the equation exceeds four, z is invariant under a cyclic substitution of three letters. The relation between the roots $x_1 = \psi(x_1, x_2, x_3, x_4, x_5)$ must be an identical equation. This is impossible since the left member is altered by (123) and the right member is not.

The substitution theory as applied by Wantzel is reminiscent of that of Ruffini, yet is far from identical with it. Wantzel stands no closer to Ruffini in this second part of the proof than he stands to Abel in the first part. It is what one might have expected Wantzel to contribute, after "meditating on the researches of the two geometers." The proofs of Ruffini and Wantzel differ altogether in the first part. Hence the claim put forth by Burkhardt and Bortolotti, that Wantzel's proof is the same as Ruffini's, is wholly unsupported by the facts as regards the first part and is too sweeping as regards the second part.

The "Irreducible Case" in Cubics.

Quite forgotten are the proofs given by Wantzel of three other theorems of note, viz., the impossibility of trisecting angles, of duplicating cubes, and of avoiding the "irreducible case" in the algebraic solution of irreducible cubics. For these theorems Wantzel appears to have been the first to advance rigorous proofs. To be sure, Paolo Ruffini, in his booklet of 1813 (pages 54–57), had put forth a proof that the "irreducible case" is unavoidable, but in the absence of satisfactory demon-

stration that the root must take a certain form, it is open to objection. Wantzel proves this theorem at the end of his paper of 1843, previously referred to. He writes a root of the irreducible cubic in the form of an entire radical function of the nth kind (espèce), $x = A + Bu + \cdots + Mu^{m-1}$, where $u = \sqrt[m]{a}$, a is a radical function of the $(n-1)$th kind, while $A, B, \cdots, M$ may be such functions of the nth kind but are of a degree inferior to x. Roots of imaginary numbers do not occur in x. He shows that B may be taken ± 1. Substituting for u in x the roots of the irreducible equation $u^m - a = 0$, he finds that $m = 3$ or 2. If $m = 3$ and $\alpha^3 = 1$, he proves that

$$x + \alpha x_1 + \alpha^2 x_2 = \pm 3u.$$

Since, by supposition, x, x_1, x_2, u are all real, it follows that $x_1 = x_2$, which is impossible since the given cubic is irreducible. He shows that $m = 2$ is likewise impossible. Hence the "irreducible case" cannot be avoided. Nearly half a century later proofs were given of this theorem by Mollame (1890), Hölder (1891), and Kneser (1892).

Duplication of Cubes and Trisection of Angles.

These problems are taken up in Wantzel's article in *Liouville's Journal*, volume 2 (1837), pages 366–372, on the "means of ascertaining whether a geometric problem can be solved with ruler and compasses." He shows first that problems solvable by ruler and compasses can be solved algebraically by a series of quadratic equations, (S) $x_i^2 + A_{i-1}x_i + B_{i-1} = 0$ $(i = 1, 2, \cdots, n)$ where A_0 and B_0 are rational functions of given numbers $p, q, \cdots$, and where A_i and B_i are rational functions of $x_i, x_{i-1}, \cdots, x_1, p, q, \cdots$. Secondly, he shows that, if in A_{n-1} and B_{n-1} we substitute in succession the two values of x_{n-1} obtained by solving $x_{n-1}^2 + A_{n-2}x_{n-1} + B_{n-2} = 0$, and if thereupon we multiply together the two resulting expressions for the left member of $x_n^2 + A_{n-1}x_n + B_{n-1} = 0$, we obtain an equation of the fourth degree in x_n. Its coefficients are rational functions of $x_{n-2}, x_{n-3}, \cdots, x_1, p, q, \cdots$. Repeating this process by eliminating successively $x_{n-2}, \cdots, x_1$, he obtains one equation of the degree 2^n in x_n, the coefficients of which are rational functions of $p, q, \cdots$. Thirdly, he shows that an equation of the degree 2^n, resulting from the least possible number of quadratic equations necessary to solve

a given problem by ruler and compasses, is irreducible. Fourthly, he considers tests for determining whether a proposed irreducible equation of the degree 2^n can be solved by a series of square root extractions. This is done by equating the coefficients of the proposed equation with those of the general equation of the degree 2^n obtained from the system (S). If thereby the coefficients of the quadratic equations (S) can be obtained by the extraction of square roots, but of no higher roots, then the proposed equation can be solved in that manner.

Wantzel then remarks that the equation $x^3 - 2a^3 = 0$, arising in the problem of the duplication of a cube, is irreducible, but is not of the degree 2^n; hence the cube cannot be doubled in volume by a construction with ruler and compasses. He draws the same conclusion for the equation $x^3 - \frac{3}{4}x + \frac{1}{4}a = 0$, on which depends the trisection of an angle. His own words are: "Cette équation est irréductible si elle n'a pas de racine qui soit une fonction rationnelle de a et c'est ce qui arrive tant que a reste algébrique; ainsi le problème ne peut être résolu en général avec la règle et le compas.' Il nous semble qu'il n'avait pas encore été démontré rigoureusement que ces problèmes, si célèbres chez les anciens, ne fussent pas susceptibles d'une solution par les constructions géométriques auxquelles ils s'attachaient particulièrement."

Saint-Venant admits Wantzel's claim of priority and adds that somewhat later Charles Sturm simplified the proofs but did not publish them. So far as now known, Wantzel's priority in publishing detailed, explicit and full proofs of the impossibility of doubling cubes, of trisecting angles and of avoiding the "irreducible case" in the cubic is incontested.

COLORADO COLLEGE,
COLORADO SPRINGS.

ADDITIVE FUNCTIONS OF A POINT SET.

Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire. Par C. DE LA VALLÉE POUSSIN. Paris, Gauthier-Villars, 1916. viii + 154 pp.

THE general notion of an additive function of a point set is one of the most important introduced by the present French school of mathematicians. It is due to H. Lebesgue. An additive function of a point set is one whose value on a sum of sets is the sum of its values on each term. These terms, the number of which may be infinite, have in pairs no common element. The volume under review (one of the Borel monographs) is devoted to the theory of additive functions of a point set; it contains the matter of lectures delivered at the College of France between December, 1915, and March, 1916. The author had already treated the problem in his Harvard lectures, which were partially published in 1915 in the *Transactions of the American Mathematical Society*. Some of the results had also appeared in volume II of his Cours d'Analyse, third edition. In the present volume the treatment is rendered distinctly more complete and satisfying through the use of new methods and the derivation of new results.

In these lectures we have a careful analysis of the notion of additivity and the derivation of the consequences, singularly precise and interesting, which follow from the sole property of a function implied in this notion.

The simplest and earliest known additive function of a point set is its measure. The definition was given by Borel in 1898. It furnishes the point of departure for the entire theory of additive functions. It is this function which is considered in the beginning of the first of the three (nearly equal) parts of the present monograph. (Lebesgue integrals are treated in the latter portion of this part.) The measure is a non-negative function whose value is given on certain point sets called elementary figures and is then to be defined on other sets in such way as to satisfy the requirement of being additive. The point sets on which this requirement is satisfied are then the measurable sets.

If a function possesses the property of being additive only

for a finite number of terms it is said to be additive in a restricted sense; if for an enumerable infinitude of terms it is said to be additive in a complete sense. The essential progress obtained by Borel and Lebesgue in the theory of measure consists in their having realized additivity in the complete sense. The first idea of this theory is due to Borel. The proper work of Lebesgue commences only with definite integrals.

The question of measure from the point of view of Borel's contribution gives rise to the following difficult problem: "a function being given on certain particular point sets, such as the elementary domains, does there exist an additive function which coincides with the preceding on these domains?" It is this fundamental question which is resolved in the second part of these lectures. One reaches the conclusion that there is complete equivalence, under the condition of continuity, between an additive function of a point set and a function $f(x)$ of bounded variation.

A function $g(e)$ of a point set e is said to be continuous if its value approaches zero with the diameter of e, and to be absolutely continuous if its value approaches zero with the measure of e. Among the additive functions of a point set the most important are those which are absolutely continuous. The indefinite integrals of Lebesgue possess these two properties of additivity and absolute continuity and are characterized by them. The derivatives of absolutely continuous functions of a point set are studied (also in the second part) by the new method of networks (réseaux) already utilized in the Harvard lectures. We return later to a discussion of this method.

The questions mentioned in the foregoing paragraphs are throughout of a metric nature. Questions of a more exclusively descriptive kind, but intimately related to the foregoing, are treated in the third part. Here the author takes up the separation of functions into the successive classes of Baire, simplifying and completing the exposition of this interesting theory by the introduction of new methods and the addition of new results. The relation of Baire's classification to the other matters treated in the book is brought out by showing the identity of the class of functions measurable in the sense of Borel and the class of functions contained in Baire's classes of all orders.

Thus the questions treated in this work belong to the recent theory of functions of which the founders are Borel, Baire and Lebesgue. In the course of his profound researches Baire has set off a real functional domain which appears to suffice for all the needs of analysis and seems to be such that every generalization is condemned beforehand to be vain and sterile. The functions of this domain possess precise common properties. The general methods of analysis are applicable to them; and their theory, already rich in results, may be considered as the *general theory of functions of a real variable.*

Thus we have here a fundamental advance from the general point of view of the philosophy of mathematics. In this progress the work of Lebesgue has been central. More than any other, Lebesgue has contributed to put in evidence the unity of this theory of functions of real variables and to beautify it by bringing out that aesthetic character which delights the artistic spirit of the pure mathematician. The author has achieved his aim in the book if the reader recognizes this character in his pages.

Such are the general features of this monograph in which there is much to please one and nothing to annoy. It remains to give a brief account of the novelties in the author's methods and results.

In the first part, which deals with measurable sets and the integrals of Lebesgue, the extent of novelty is not so great as in the other two parts. Here the author treats general notions about point sets, measure of sets and measurable functions, and Lebesgue integrals, devoting a chapter to each of the three topics. On page 10 is given a generalization of the usual definitions of interior, exterior, and frontier points of a set (see also page 105). But the chief novelty to which attention may be called is in the notion of the characteristic function of a point set, a notion already utilized by de la Vallée Poussin in his *Transactions* memoir.

Let E be a point set in a space $x, y, \cdots$ of one or more dimensions and let CE denote the set complementary to E in such a space. Then the characteristic function or the characteristic of the point set E is the function $\varphi(x, y, \cdots)$ which has the value unity at each point of E and the value zero at each point of CE. It is clear that $1 - \varphi$ is the characteristic of CE. If $\varphi_1, \varphi_2, \cdots$ are the characteristics in order of $E_1, E_2, \cdots$, these sets having in pairs no common elements, then

$\varphi_1 + \varphi_2 + \cdots$ is the characteristic of $E_1 + E_2 + \cdots$. Also, $\varphi_1\varphi_2\cdots$ is the characteristic of $E_1E_2\cdots$ without condition on $E_1, E_2, \cdots$. To find the characteristic of the sum $E_1 + E_2 + \cdots$, when no condition is put on the component sets, observe that this sum is the complement of the set $CE_1 \cdot CE_2 \cdot CE_3 \cdots$; hence its characteristic is

$$1 - (1 - \varphi_1)(1 - \varphi_2)(1 - \varphi_3)\cdots.$$

Through the use of characteristic functions the problem of infinite sums and products of point sets is readily reduced to that of the passage to a limit so that certain fundamental properties of such sums and products may at once be read off by means of corresponding properties of limits.

It is not difficult to show (see page 36) that the characteristic function of a point set which is measurable in the sense of Borel belongs to one of Baire's classes. This leads to a natural classification of point sets which are measurable in the sense of Borel, the class number of the set being the same as that of the corresponding characteristic function in the classification of Baire. Certain relations among the classes of point sets then follow at once from corresponding relations among the classes of Baire.

In approaching the theory of measure in the second chapter, de la Vallée Poussin develops first of all the theory of measure for enumerable sums of intervals and closed sets of points and then through the aid of this theory defines the measures of more general sets. In this connection it is of interest to note that Bliss (this BULLETIN, volume 24 (1917), pages 1–47) has returned to the more direct methods of Borel and Lebesgue which found the theory of measure entirely upon the measurability of enumerable sets of intervals. Through aid of the improved methods of de la Vallée Poussin, Bliss is able to approach the subject in a way which is especially concise and clear. At the same time he establishes without additional complication the foundations of the theory of the positive additive functions of a point set, of which the measure function is a special case. The reader of de la Vallée Poussin will find it useful to compare the improved exposition of Bliss.

The second part of the monograph deals in general with additive functions of a point set, its three chapters having in order the following titles: general notions concerning derivatives and networks; absolutely continuous and additive func-

tions of a point set, indefinite integrals; additive functions of a normal set. This part is rich in novelty. One meets here the term symmetric derivatives (page 59), the particularly fruitful notion of networks (pages 61 ff.) and of derivatives on a network, and again (page 74) the major and minor functions of a summable function $f(x)$ which de la Vallée Poussin first introduced in 1909 in the second edition of volume I of his Cours d'Analyse. The greater part of Chapter VI is based on the author's *Transactions* memoir; the exposition is simplified and the results are frequently generalized or rendered more precise.

The new method of networks already employed by de la Vallée Poussin in his Harvard lectures is now rendered much more useful through the notion of conjugate networks. By aid of this, certain demonstrations are made in a simpler and more natural manner than before and the form of the exposition is improved in many respects. The novelty associated with this method constitutes one of the two central features of the monograph and is perhaps the most interesting illustration in it of the author's insight into the problems with which he is dealing. Fortunately a clear, excellent and readable connected exposition of these new ideas is given on pages 61–66, so that it is unnecessary for us to attempt a summary here in our limited space. The reader who wishes to follow up some of the main uses of the method elsewhere in the monograph may consult pages 67–82 and 96–104.

The second central feature of the monograph is to be found in the third part in the treatment of Baire's classification of functions. In his thesis (1899) Baire demonstrated the following theorem: A necessary and sufficient condition that a discontinuous function f shall be of class 1 on a bounded and perfect set P is that it shall be pointwise discontinuous on every perfect set Q contained in P. This is called Baire's theorem. It is intimately associated with the problem of Baire, namely: Given a function f of class 1, to construct a sequence of continuous functions $f_1, f_2, \cdots$ such that f_n approaches f as n approaches infinity.

Baire proved the last preceding theorem by the direct process of resolving the corresponding problem. The first demonstration was confined to a function of a single variable. Lebesgue (1899) proved the theorem for a function of several variables by a process of reduction to the case of a single

variable. Baire extended his original method so as to render it applicable to functions of several variables. Finally he gave a systematic exposition of his theory in 1905 in his Leçons sur les Fonctions discontinues.

The methods of Baire make appeal to the transfinite. Lebesgue* has given demonstrations of Baire's theorem not requiring the intervention of the transfinite. He states the condition in the form: A necessary and sufficient condition that a discontinuous function f shall be of class 1 on a bounded and perfect set P is that the set of points where $f > A$ and the set where $f < A$ shall be sums of closed sets for every A. The methods of Lebesgue do not afford a solution of Baire's problem.

In the present monograph we have a new demonstration of Baire's theorem and a new and simple resolution of Baire's problem. The method of exposition is a sort of synthesis of those employed by Lebesgue and Baire. It completes those of Lebesgue and, by the introduction of an auxiliary theorem, notably simplifies that of Baire.

In the theorems of Baire and Lebesgue we have two very different expressions of the condition that a function shall be of class 1. If one seeks to generalize these theorems and to obtain analogous distinctive characteristics of functions of general class α ($\alpha > 1$), he finds that Baire's form generalizes only imperfectly (see page 144) while an adequate and complete generalization of Lebesgue's form is available (see page 141). Lebesgue's method of treatment is here followed in the main, with improvements in certain parts and the introduction of some novel results.

For the statement of the generalization one needs to employ the notion of sets O and F of Lebesgue with respect to a bounded and perfect set P. De la Vallée Poussin puts these definitions in the following form: *A set E is an F of class α, if there exists a function θ, defined on P and of class $\leqq \alpha$, such that the set E coincides with the set of points of P on which $\theta = 0$; a set E is an O of class α, if there exists a function θ, defined on P and of class $\leqq \alpha$, such that the set E coincides with the set of points of P on which $\theta \neq 0$.*

The generalized condition of Lebesgue may now be stated in the following form: A necessary and sufficient condition

* Borel's Leçons sur les Fonctions de Variables réelles et leur Représentation par les Séries de Polynomes, Note II; *Bulletin de la Société mathématique de France*, 1905.

that a function f (finite or not) shall be of class $\alpha > 0$ on P is that for every constant a the set where $f > a$ and that where $f < a$ shall be sets O of class α and shall not be both of class less than α; or, what amounts to the same thing, that the (complementary) set where $f \leqq a$ and that where $f \geqq a$ shall be sets F of class α and not both of class less than α.

The monograph closes with a demonstration (at bottom agreeing with that of Lebesgue but expressed in somewhat more elementary form) of the theorem which asserts the existence of functions of all of Baire's classes.

In view of the numerous definitions of integration which have been given and the present state of unrest in this range of ideas, one desires to consider Lebesgue integrals not merely in themselves but also in their relation to other types of integrals. Such a comparative treatment does not fall within the scope of de la Vallée Poussin's monograph. Fortunately a careful analysis of this sort is now (readily) accessible in Hildebrandt's recent paper (this BULLETIN, volume 24 (1917–1918), pages 113–144, 177–202).

In the monograph under review and in the two articles by Bliss and by Hildebrandt already referred to we have recent treatments of the general problem of integration which supplement and complete each other in a useful way. De la Vallée Poussin develops the theory of Lebesgue integrals in intimate association with several fundamental matters to which at bottom they are closely related, so that through this monograph one may obtain a view of this domain of mathematics in its proper relation to other fields and in appropriate perspective as to its own parts among themselves. Bliss gives a particularly compact, but at the same time clear, development of the theory of Lebesgue integrals and the intimately associated theory of measure, the march of ideas in his treatment being singularly direct and leading the reader to the goal without loss of energy due to indirect processes and hence with a minimum of effort. Hildebrandt's purpose is to discuss briefly the several definitions of integration and particularly to consider their relations one to another. He gives careful attention to the distinctive features of each definition, to the question of equivalence, and to the problem of further generalization.

Equivalences among integrals are of two types. The first may be called complete equivalence; it is one in which the

two symbols of integration operate upon the same class of functions and give the same integral value for the same function. The other is in reality a pseudo-equivalence expressed in the fact that an integral of one kind may be transformed into an integral of another kind, the functions integrated in the two cases being different. A careful and instructive analysis of these equivalences is given by Hildebrandt.

To one result of this analysis it is desirable to have attention sharply directed. The Stieltjes integral seems destined to play in the future a rôle of central importance in processes of integration and summation. The Lebesgue integral when introduced received almost immediate attention and recognition and found its way rapidly into the main current of mathematical thought; but the Stieltjes integral has been singularly neglected notwithstanding its inherent simplicity and naturalness. Through the summary of its properties given by Hildebrandt and the applications mentioned one is convinced of its central importance and is led to expect it to assume a new place in mathematical thought. In this connection it is of particular interest to note also Hildebrandt's extension of the Stieltjes integral modelled on the Lebesgue extension of the Riemann integral.

We conclude with the following list of misprints in de la Vallée Poussin's monograph: page 20, last line, write $m(F_1 + F_2) = mF_1 + mF_2$; page 28, second theorem, write $(f \geqq A)$ instead of $(f \geqq a)$; page 34, line 16, write $\omega 2$ instead of ω_2; page 96, end of second paragraph, write Df instead of DF; page 126, line 9 below, write "Une" instead of "Uue"; page 133, line 2, write "de classe $\leqq \alpha$."

R. D. CARMICHAEL.

SHORTER NOTICES.

Leçons de Mathématiques Générales. Par L. ZORETTI. Paris, Gauthier-Villars, 1914. 8vo. xvi + 753 pp.

Exercices numériques de Mathématiques. Par L. ZORETTI. Paris, Gauthier-Villars, 1914. 8vo. xv + 125 pp.

THE author of this text and its accompanying set of exercises is well qualified for the task. Formerly instructor in the

lycée, contributor to the theory of functions by several important memoirs, professor of mechanics, he is fully equipped to add to several already existing treatises on mathematics in general another of the particularly successful French treatises of this kind. His special object was to write a book that would furnish the necessary knowledge for the most widely different classes of students, whether they intended to enter experimental science, engineering, the ranks of teaching, or some of the professions. His goal is that of applications, and everything is included that might be useful for practical mathematics, everything is excluded that cannot at once justify its applicability elsewhere.

This program, to be realized in a manageable space, requires a presentation which is as intuitive as possible, but still attacking with all frankness demonstrations which must be given, and further considering carefully the conditions under which the results that are arrived at may be applied, and the significance of these results. Exactness in definition is also necessary in order that the student may clearly understand what he is doing. The student is thrown upon his own resources, particularly in the exercises, many of which demand careful construction and measurement by the student himself. We might say that a certain reasonable amount of laboratory mathematics is insisted upon in order that the student may learn not only to apply mathematics to data given in some way, but to derive the data to which the mathematics is to be applied.

The first 157 pages are devoted to a survey of analytical geometry, including the theory of vectors. This grouping of the subjects is perfectly proper inasmuch as analytical geometry is merely a rather primitively developed vector analysis. The first chapter defines vector (directed segment), the equivalence of vectors which are on the same axis, and the equipollence of vectors which are on parallel axes. Couple is included, as well as opposed vectors. Addition and subtraction of vectors on the same axis is defined, as well as the same processes for vectors whose axes are concurrent. These notions are extended to directed arcs and angles. The theorems of projection follow naturally.

The next chapter defines and illustrates the different kinds of coordinates for surfaces and for space. The method of the book is very well shown in the way in which the systems of

curvilinear coordinates are defined and their uses indicated. This kind of coordinate is seldom mentioned in the beginning of the usual texts. The elementary formulas for a line are developed, and the determination of mean distances for n points is introduced as an example, thus putting the student in touch at once with possible uses for his knowledge. The coordinates of a vector are considered next, followed by the determination of any figure, bringing in at the same time Euler's angles, and degrees of freedom. The determination of lines on surfaces, and surfaces themselves follow. Then the methods of changing the system of coordinates are developed. These ideas are then to be applied to such exercises as the following: Choose proper parameters for the following figures: segment of a line located in a given plane; circle in space; given cone of revolution, the vertex in a given plane; circle tangent to a given line in space; coordinates of a point on a given line or surface.

It seems to the reviewer that such a method as this marks a distinct advance over the usual one of explaining the rectangular and polar coordinates followed by a few perfunctory plottings, and then a development of formulas for the line and plane. The student is generally supposed to seize instantly the idea of coordinates and their significance, and to be able to reason by their means with no further study of them. Their significance as parameters or numbers which may arise in an infinity of ways, different sets being useful for different purposes, is seldom pointed out, and the student is generally unable to state his problem in any other way than by the conventional methods of x, y, z or r, θ, φ. When he comes to integration in space he is usually helpless.

Chapter three develops the fundamental formulas of the line and plane. Chapter four considers the theory of vectors, moments, resultants, resultant moments, equivalent systems, invariants, central axis, couples, coplanar vectors, and parallel vectors. The student is early equipped with useful knowledge for statics. Two chapters follow on circles and conics. In the latter the approximate construction by circular arcs is described. In the manual of exercises the second series contains other methods of drawing these curves, with some original theorems on them easily deduced from the methods. Chapters seven and eight consider quadrics and some of the common curves and surfaces.

The second part of the work is given up to algebra, theory of functions, and derivatives. It occupies 342 pages, the greater part being concerned with functions. The algebraic topics cover complex numbers and their representation, the binomial theorem, determinants, and elimination. A chapter is devoted to infinite series, with the convergence tests of D'Alembert, Cauchy, and the rule that is derived by comparison of the series with the series Σn^{-p} where $1 < p$. These are characterized for practical purposes as follows: The last reveals only slowly convergent series, the second is not easy of application, so that the rule of D'Alembert is about the only practical one.

The notion of function is based upon dependence, but the illustrations are of natural dependence and not of the artificial type often seen in such examples. The author then defines, and considers the uses of, entire functions, simply periodic functions (a vastly better name than harmonic functions), the exponential function, circular functions as exponential, hyperbolic functions, logarithmic function, inverse circular functions, inverse hyperbolic functions, with some closing remarks on graphs when the quantities represented by x and y are not of the same kind.

In discussing differentiation examples are freely used to develop some very fundamental notions, often not mentioned in the beginning works on calculus studied by engineers and other practical students. For instance, it is shown by a wave curve that lies in a very narrow strip, that while functions which have derivatives which lie close together do not differ much themselves, on the other hand functions which lie close together may have derivatives very far apart. For this reason integration never has the hard conditions to meet which restrict derivation. Another instance of departure in a useful way from other elementary presentations is in the definition and use of the directed derivative of scalar and vector point functions, notions much more useful than a large set of complicated algebraic formulas for differentiation. A chapter of variations of functions presents several well known curves and some of their peculiarities. A short chapter on development in series contains the essentials. Derivatives are applied to curves, surfaces, and movements, including relative motion. A chapter is given to theory of equations and their solutions, and a chapter to numerical and graphical calculation.

The third part of the work covers 248 pages, and considers the integral calculus and its applications. The integral is defined as the limit of a sum, making the definite integral properly the foundation of the whole subject. The connection with the primitive of the simple differential equation $dy = f(x)dx$ is immediately brought out, and some of the formulas of differentiation, as $d \cdot uv = vdu + udv$, are interpreted in integration, this one leading to integration by parts. The calculation of mean values is given as an example of the use of the definition. A chapter follows on general methods of finding indefinite integrals. Chapter three extends the notion in some directions, as that of improper integrals, line integrals, double and triple integrals, leading up to the formulas of Green and Ostrogradsky, although singularly the complete formula of Stokes is omitted at the point where it would have naturally come in. A chapter is given to elliptic functions, one to Fourier series, one to geometric applications, one to applications to mechanics. In the latter the notions of vector fields, curl, divergence, flux, vector lines and tubes, work, circulation, level surfaces, and potential are brought in. The approximate calculation of integrals has a chapter. The last two chapters consider differential equations, total and partial, and though quite elementary, the author has nevertheless given the student a good working basis. The accompanying problems of the manual of exercises reenforce the text materially.

The author states that he considers the needs of the student who has to study by himself, with little or no assistance. This no doubt accounts for the very plain treatment, and would suggest that in general a text written for students will be clearer than one written for the use of teachers. Of course a certain amount of rigor in books written for teachers is demanded, but rigor does not always lead to usable knowledge. Professor Zoretti is to be congratulated on his success.

JAMES BYRNIE SHAW.

College Algebra with Applications. By E. J. WILCZYNSKI. Edited by H. E. SLAUGHT. Boston, Allyn and Bacon, 1916. xx+507 pp.

THIS algebra, unlike the traditional college algebra, possesses unity, the centralizing theme being the function concept. The book opens with an excellent chapter on the number sys-

tem of algebra, which is developed by means of the geometry of directed line segments. This chapter is followed by chapters on linear functions and progressions, quadratic functions and equations, integral rational functions of the nth order, fractional rational functions, irrational functions, the exponential function and logarithms, linear functions of more than one variable, permutations and combinations, probability, determinants, quadratic functions of two independent variables and simultaneous quadratic equations, sequences and series with a finite number of terms, limits and infinite series. There is an appendix which contains a short table of logarithms and a mortality table.

Throughout the book much emphasis is laid on the applications of algebra to problems in physics and chemistry. Such topics as the measurement of time, length and mass; the theory of the vernier and slide rule, logarithmic paper and scales; the notions of velocity, acceleration, density, specific gravity, force, uniform motion, etc., are fully discussed in the text, thus making it unnecessary for a student to have had a course in physics and chemistry.

The author's explanations are very lucid, but I wish to call particular attention to his treatment of complex numbers, partial fractions, and the notorious stumbling block "mathematical induction," where by means of some well-chosen examples he drives home the fact that both parts of the proof are equally important.

The author has introduced some analytics and some calculus in order to weave his subject matter about the function concept. He defines the slope of a line, finds the equations of lines, defines and derives the derivative of a rational integral function, applies derivatives to Taylor's expansion, maxima and minima, the finding of approximate irrational roots (Newton's method), etc.

Throughout the book appear many historical references which add greatly to the interest of the subject matter.

The reviewer believes that had the author omitted a few of the interesting but less important topics, and had he inserted more exercises, but preserved the unity which he has so carefully worked out, his book would serve better the needs of the average American college. This is an admirable book of reference and should be owned by every progressive teacher of mathematics.

F. M. MORGAN.

Annuities and Amortization Tables. By PIERRE ZALDARI. New York, Bankers Encyclopedia Co., 1917. 350 pp. Price $10.00.

THIS notice is written after reading the page proofs of the book that was to come from the press on December 15.

The preface states that "the preparation of this book was suggested to the writer primarily by the more or less recent development of what may be called foreign financing in this country, and the greatly increased importance in the position which America has assumed as a factor in the general scope of international banking; and some of these tables and problems were prepared to meet the specific demands of the Federal Farm Loan Act in that section which provides that the Farm Loan Board 'shall prepare and publish amortization tables.'"

The book contains 70 pages in which are propounded and solved 104 problems that embrace the whole range of the intricacies that arise in finance, covering the subjects of compound interest, annuity or annual investment, amortization, loans—issued at par, issued at a rate different from par and refunded at par, refunded above par, refunded above par with lottery privileges or premiums, the determination of the income rate and calculation of parities. There are ten chapters in this first part of the book. They are notably clear and in a form to make them readable by any one concerned, without a knowledge of mathematics beyond that given in the high school. Several of the solutions are not the ordinary ones in vogue in this country, and are quite neat.

The seven tables, only sample pages of which are before the reviewer, are to occupy 280 pages. The type and arrangement make these tables notably easy for reference. They are adequate to the solution of each of the 104 problems by a single multiplication. The tables have a wider range of usefulness than any that careful search has revealed within the covers of a single volume. They have more rates of interest than other tables, going from $\frac{1}{4}$ of 1 per cent to 10 per cent by steps of $\frac{1}{8}$ of 1 per cent. The periods are from 1 to 100. Some of the tables have no counterpart in other sets of tables. The numbers given in the tables carry from seven to eleven decimal places. Such a set of tables cannot but be received well by financiers in all lines of finance.

CHARLES C. GROVE.

NOTES.

At the Pittsburgh meeting of the American association for the advancement of science, assistant professor G. D. Birkhoff, of Harvard University, was elected vice-president of Section A, and Professor F. R. Moulton, of the University of Chicago, was reelected secretary of the Section.

The March number (series 2, volume 19, number 3) of the *Annals of Mathematics* contains the following papers: "Comparison theorems for homogeneous linear differential equations of general order," by R. D. Carmichael; "Note on a substitute for Duhamel's theorem," by H. B. Fine; "Some properties of a straight line and circle and their associated parabolas," by J. H. Weaver; "Motions in hyperspace," by C. L. E. Moore; "A definition of sense on closed curves in nonmetrical plane analysis situs," by J. R. Kline; "Covariant expansion of a modular form," by O. E. Glenn; "The intersections of a straight line and hyperquadric," by G. M. Green; "Numerical functions of $[x]$," by E. T. Bell; "Surfaces which can be generated in more than one way by the motion of an invariable curve," by L. P. Eisenhart.

The following mathematical papers appeared in volume 52 (1916–1917) of the *Proceedings of the American Academy of Arts and Sciences:* "Differential geometry of two-dimensional surfaces in hyperspace," by E. B. Wilson and C. L. E. Moore, number 6, pages 267–368; "A classification of quadratic vectors," by F. L. Hitchcock, number 7, pages 369–454.

At the meeting of the London mathematical society held January 17, 1918, the following papers were read: By P. A. MacMahon, "A method for studying any convergent series"; by G. H. Hardy, "Additional note on Dirichlet's division problem"; by J. H. Grace, "Note on a Diophantine approximation"; by K. A. Rau, "A note on a theorem of Mr. Hardy's"; by C. H. Forsyth, "Supernormal curves"; by H. Hilton and Miss D. S. Tuck, "Plane quartic curves with a tacnode."

THERE are probably many libraries throughout the country which contain the two volumes of "Correspondence of Scientific Men of the Seventeenth Century . . ." edited by Rigaud and published by the Oxford University Press in 1841. In 1862 a 36-page table of contents and index, compiled by Augustus De Morgan, was also issued (see Budget of Paradoxes). It is learned that these extra pages may still be purchased from the Oxford University Press, for forty cents.

THE following university courses in mathematics are announced for the summer session:

CORNELL UNIVERSITY (July 8 to August 16).—By Professor W. B. CARVER; Geometry for teachers, three hours.—By Professor W. A. HURWITZ: Algebra for teachers, three hours.—By Professor F. W. OWENS: Projective geometry, three hours.

UNIVERSITY OF CALIFORNIA (June 24 to August 3, at Los Angeles).—By Professor C. A. NOBLE: Analytic geometry, five hours; Mathematics for teachers, five hours.—By Professor VIRGIL SNYDER (of Cornell University): Calculus, five hours; Higher geometry, five hours.

UNIVERSITY OF CHICAGO (June 17 to August 31).—By Professor E. H. MOORE: Differential equations in general analysis (first half), four hours.—By Professor G. A. BLISS: Calculus of variations, four hours; Higher plane curves, four hours.—By Professor L. E. DICKSON: Solution of numerical equations (first half), four hours; Determinants and symmetric functions (second half), four hours; Algebraic invariants, four hours.—By Professor H. E. SLAUGHT: Definite integrals, four hours.—By Professor A. C. LUNN: Units and dimensions, four hours; Electromagnetic theory, four hours.—By Professor J. W. A. YOUNG; Topics in geometry, four hours.—By Professor R. G. D. RICHARDSON (of Brown University): Theory of functions of a complex variable, four hours.—By Professor W. H. ROEVER (of Washington University): Descriptive geometry, five hours.

UNIVERSITY OF ILLINOIS (June 18 to August 9).—By Dr. A. J. KEMPNER: Advanced algebra.—By Professor J. B.

SHAW: Differential equations; Vector methods.—By Professor C. H. SISAM: Solid analytic geometry; Differential geometry.

UNIVERSITY OF WISCONSIN (June 24 to August 2).—By Mr. R. E. MOORE: Differential equations, introductory course, five hours.—By Professor E. B. SKINNER: Differential geometry, three hours.—By Professor H. W. MARCH: Mechanics of a rigid body, five hours; Theory of probability, three hours.—By Dr. T. M. SIMPSON: Equations of the third and higher degrees, three hours.—By Professor E. B. VAN VLECK: Theory of the linear substitution in n variables, three hours; A survey of geometry, algebra, and trigonometry, treating of their development and history, five hours.

THE Academy of Sciences of Vienna has awarded the Baumgartner prize to Professor A. EINSTEIN.

PROFESSOR PAUL PAINLEVÉ has been elected president of the Paris academy of sciences.

MR. C. HANNEVIG has donated 150,000 crowns as a memorial to N. H. ABEL. The income of the fund is to be used to further mathematical research in Norway.

PROFESSOR L. CRELIER has been promoted to a full professorship at the University of Bern.

THE technical school at Vienna has conferred honorary doctorates on Professor R. MEHMKE, of the technical school at Stuttgart, and on Professor S. FINSTERWALDER, of the technical school at Munich.

PROFESSOR M. DISTELI, of the technical school at Carlsruhe, has retired, on account of ill health. Professor H. MOHRMANN, of the school of mines at Clausthal, has been appointed his successor.

PROFESSOR H. HAHN has been promoted to a full professorship of mathematics at the University of Bonn. DR. A. BECK has been appointed associate professor of mathematics.

PROFESSOR E. SCHMIDT, of the University of Breslau, has been appointed professor of mathematics at the University of Berlin.

THE following university and college teachers of mathematics have recently entered the national military service:

Mr. J. E. DAVIS, of Pennsylvania State College, has joined the national army. Mr. J. J. TANZOLA, of the U. S. Naval Academy, has joined the national army. Mr. H. M. TERRILL, of Columbia University, has been made ensign in the naval reserve. Mr. A. L. WECHSLER, of Columbia University, has joined the national army. Professor V. H. WELLS, of the University of Pittsburgh, has been made lieutenant in the science and research division of the signal reserve corps. Mr. FREDERICK WOOD, of the University of Wisconsin, has been appointed lieutenant in the field artillery.

THE Normal medal of the American Society of Civil Engineers has been awarded to Mr. B. F. GROAT, formerly professor of mathematics and mechanics at the University of Minnesota and now a consulting engineer of Pittsburgh, for his paper "Chemihydrometry and its application to the precise testing of hydro-electric generators." The theoretical considerations involved are largely of a mathematical nature.

PROFESSOR A. G. CARIS, of the department of mathematics, has been appointed acting president of Defiance College, Defiance, Ohio.

PROFESSOR P. F. SMITH has been made acting director of the Sheffield Scientific School, Yale University, during the absence of Director R. H. Chittenden on government service.

MR. M. F. JORDAN and Mr. Q. STAUFFER have been appointed instructors in mathematics at the University of Maine.

PROFESSOR LÉON AUTONNE, of the University of Lyons, died January 12, 1916, at the age of fifty-seven years.

PROFESSOR G. FROBENIUS, of the University of Berlin, died August 3, 1917, at the age of sixty-seven years.

PROFESSOR FRANZ LONDON, of the University of Bonn, died February 17, 1917, at the age of fifty-four years. Mr. ALFRED HAMBURGER has presented the University with an endowment of 30,000 marks for a foundation for mathematical research, in memory of Professor London.

PROFESSOR CHRISTIAN HORNUNG, of Heidelberg University, Tiffin, Ohio, died January 31, 1918, at the age of seventy-three years.

PROFESSOR A. T. G. APPLE, of the department of mathematics and astronomy of Franklin and Marshall College, died February 5, 1918, at the age of fifty-eight years.

DR. E. W. PONZER, assistant professor of applied mathematics at Stanford University, died December 20, 1917.

DR. E. A. ENGLER, formerly professor of mathematics and descriptive geometry at Washington University, St. Louis, and for ten years president of Worcester Polytechnic Institute, died January 16, 1918, at the age of sixty-one years. Professor Engler had been a member of the Society since 1892.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAM (C.). See DESCARTES (R.).

BOREL (E.). Leçons sur les fonctions monogènes uniformes d'une variable complexe. Rédigées par G. Julia. (Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Emile Borel.) Paris, Gauthier-Villars, 1917. 8vo. 12+166 pp. Fr. 7.50

BOWSER (E. A.). College algebra for the use of academies, colleges and scientific schools. 12th edition. Boston, Heath, 1917. 8vo. 14+540 pp. $1.60

DESCARTES (R.). Œuvres de Descartes publiées par C. Adam et P. Tannery sous les auspices du Ministère de l'Instruction publique. Paris, Cerf. Tomes 1–11: Œuvres, 1893–1909. Tome 12: Vie et œuvres de Descartes, étude historique par C. Adam, 1910. 20+646 pp. [Tome 13]: Supplément, index générale, 1913. 108 pp. 4to. Fr. 310.00

JULIA (G.). See BOREL (E.).

TANNERY (P.). See DESCARTES (R.).

II. ELEMENTARY MATHEMATICS.

DELBRIDGE (C. L.). Delbridge wonderful interest tables, 1 day to 365 days. . . . St. Louis, Delbridge, 1916. Folio. 371 pp. $20.00; leather, $25.00

DUDENEY (H. E.). Amusements in mathematics. London, Nelson, 1917. 8+258 pp. 3s. 6d.

FORSYTH (C. H.). Logarithms and anti-logarithms. Ann Arbor, George Wahr, 1917. 46 pp. Cloth. $0.75

JANNELLI (G.). Studio sulla trisezione dell'angolo. Palermo, tipografia Pontificia, 1917. 4to.

LENNES (N. J.). See SLAUGHT (H. E.).

MEAD (C. DEW.). Experiment in the fundamentals; giving the results of tests made in the Cincinnati schools with two kinds of practice material. (School efficiency monographs.) Chicago, World Book Company, 1917. 8vo. 14+54 pp. $0.60

SHORTHOSE (—.). Commercial ready reckoner. 9th edition. London, Allen and Unwin, 1916. 4to. 5s.

SLAUGHT (H. E.) and LENNES (N. J.). Plane geometry with problems and applications. Revised edition. Boston and Chicago, Allyn and Bacon, 1918. 8vo. 8+310 pp.

WARNE (—.). Handy ready reckoner, with over 38,000 calculations. London, Warne, 1916. 18mo. 240 pp. 7d.

III. APPLIED MATHEMATICS.

ALGRIN (—.). Eléments de mécanique générale et appliquée. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 533 pp.

ANGUS (R. W.). The theory of machines. 2d edition. New York, McGraw-Hill, 1917. 340 pp. $3.00

AUSTIN (F. E.). Examples in battery engineering. Hanover, N. H., F. E. Austin, 1917. 90 pp. Cloth. $1.25

BOUASSE (H.). Construction, description et emploi des appareils de mesure et d'observation. Paris, Delagrave, 1917. 8vo. 629 pp. Fr. 25.00

BOYD (J. E.). Strength of materials. 2d edition. New York, McGraw-Hill, 1917. 380 pp. $3.00

BRADLEE (S. G.). Stock brokers' daily balance interest table, for rapid computation on a 360 day basis . . . adapted to all classes of business where interest is figured on any amount up to $1,000,000 at every quarter per cent from 1 to 7 inclusive. Boston, Financial Publishing Company, 1916. 8vo. 95 pp. $7.50

CHRISTIE (C. V.). Electrical engineering. The theory and characteristics of electrical circuits and machinery. 2d edition. New York, McGraw-Hill, 1917. 550 pp. $4.00

COX (D. W.). Cox's tariff, tonnage and price extensions, from pounds to gross tons, computed for every hundred pounds from 100 to 160,000, at every five cents from five cents to five dollars fifty cents per gross ton of 2,240 pounds. Harrisburg, Pa., Charter Publishing Company, 1917. 4to. 222 pp. $20.00

DESMOLLIENS (M.). Cours de langue anglaise à l'usage des élèves ingénieurs. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 444 pp.

EIFFEL (G.). Nouvelles recherches sur la résistance de l'air et l'aviation faites au laboratoire d'Auteuil. Paris, Dunod et Pinat, 1914. 4to. 8+406 pp. Atlas, 67 pp.+39 plates. Reliés. Fr. 50.00

EYROLLES (L.). Formulaire mathématique et technique. 8e édition. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 16mo. 188 pp.

GABEAUD (—.). Introduction mathématique aux sciences techniques de l'ingénieur. 2e édition. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 608 pp.

GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Principes généraux, applications, exercices pratiques. Paris, Dunod et Pinat. 8vo. Tome 1, 2e édition, 1914. Tome 2, Réimpression, 1917. Tome 3, Réimpression, 1917. 386+360+182 pp. Cartonné. Fr. 4.50+4.50+2.50

HIERNAUX (G.). See GOUARD (E.).

HIGBEE (F. G.). The essentials of descriptive geometry. 2d edition. New York, Wiley, 1917. 6+218 pp. Cloth. $1.80

IZART (J.). Aide-mémoire de l'ingénieur mécanicien. Recueil pratique de formules, tables et renseignements usuels. Réimpression. Paris, Dunod et Pinat, 1917. 8vo. 28+854 pp. Cartonné. Fr. 15.00

JACQUET (A.). Aide-mémoire de l'ouvrier mécanicien. Paris, Dunod et Pinat, 1917. 8vo. 174 pp. Cartonné. Fr. 3.50

MAILLOT (L.). Le manuel du mécanicien. Paris, Dunod et Pinat, 1917. 8vo. 8+173 pp. Cartonné. Fr. 2.50

MILLS (J.). Radio communication. Theory and methods. With an appendix on transmission over wires. New York, McGraw-Hill, 1917. 11+205 pp. Cloth. $1.75

PEABODY (C. H.). Naval architecture. 4th edition. New York, Wiley, 1917. 7+641 pp. Cloth. $7.50

RICCARDO-BRAUZZI (—.). Cours d'aéronautique générale. 2e partie: Etude statique et dynamique des ballons en général. Paris, Dunod et Pinat, 1917. 8vo. 410 pp. Fr. 30.00

SEE (T. J. J.). Electrodynamic wave-theory of physical forces. Volume 1. (Bulletins 1–6 inclusive.) Lynn, Mass., T. P. Nichols, 1917. 4to. 170 pp. $5.00

SPANGLER (H. W.). Graphics. New York, Wiley, 1917. 95 pp. Cloth. $1.25

TODD (J. A.). The mechanism of exchange; a handbook of currency, banking and trade in peace and in war. Oxford, Clarendon Press, 1917. 14+255 pp.

WILLIS (S. J.). A short course in elementary mathematics and their application to wireless telegraphy. London, Wireless Press, 1917. 3s. 6d.

WOODHOUSE (H.). Text-book of naval aeronautics. London, Laurie, 1917. Royal 8vo. 35s.